Henselian Rings

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1 Henselian rings

Lemma 1 (Hensel's lifting lemma)**.** *Let A be a complete discrete valuation ring (e.g. $\mathbb{Z}_p$ or $k[\![t]\!]$) with maximal ideal $\mathfrak{m}$ and residue field k. Suppose $f \in A[x]$ is monic such that $\overline{f} \in k[x]$ factors as $\overline{f} = g_0 h_0$ with g_0 and h_0 monic and coprime (i.e. share no common factors). Then f factors as $f = gh$ with g and h monic such that $\overline{g} = g_0$ and $\overline{h} = h_0$.*

Remark 2. Such a factorization can be computed, or rather approximated, by means of an iterated process or successive approximations, such as Newton's method.

Definition 3. A *Henselian ring* is a local ring A for which the conclusion of Hensel's lemma holds.

Example 4. If A is Henselian, then so is any quotient A/I. Namely, if the factorization $\overline{f} = g_0 h_0$ lifts to A, it certainly lifts to A/I.

Remark 5. It can be shown that the lifts g and h are unique, and that they are strictly coprime, that is, $(g, h) = A[x]$.

Proposition 6. *Let A be a local ring, $X = \operatorname{Spec} A$ and let x be the unique closed point of X. The following are equivalent:*

(1) A is Henselian.

(2) Any finite A-algebra B is a product of local rings $B = \prod_i B_i$.

(3) For any étale morphism $f : Y \to X$ and a point $y \in Y$ such that $f(y) = x$ and $\kappa(y) = \kappa(x)$, there is a section $s : X \to Y$ to f, that is, $f \circ s = \mathrm{id}_X$.

(4) For any $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ and $a = (a_1, \ldots, a_n) \in k^n$ such that $\overline{f_i}(a) = 0$ for all i and $\det((\partial \overline{f_i}/\partial x_j)(a)) \neq 0$, then there exists an element $a' \in A^n$ such that $\overline{a'} = a$ and $f_i(a') = 0$ for all i.

Proof. ($1 \Rightarrow 2$) First note that, by the going-up theorem, any maximal ideal of B lies over $\mathfrak{m}$, and hence B is local if and only if $B/\mathfrak{m}B$ is local.

Suppose $B = A[x]/(f)$ for some monic $f \in A[x]$. If $\overline{f} \in k[x]$ is a power of an irreducible polynomial, then $B/\mathfrak{m}B = k[x]/(\overline{f})$ is local, so B is local. If $\overline{f}$ factors as $g_0 h_0$, then lift this factorization to a factorization $f = gh$. By the Chinese remainder theorem, $B \cong A[x]/(g) \times A[x]/(h)$ and continue recursively.

For general non-local B, there exist a non-trivial idempotent $\overline{b} \in B/\mathfrak{m}B$ which we lift to some $b \in B$. Let $f \in A[x]$ be a monic polynomial such that $f(b) = 0$, and let $\phi : C \to B$ from $C = A[x]/(f)$ be given by $\phi(x) = b$. Since C is mongenic, by the previous paragraph there exists an idempotent $c \in C$

such that $\overline{\phi(c)} = \bar{b}$. Now $e = \phi(c) \in B$ is a non-trivial idempotent in B, which yields a splitting $B = Be \times B(1-e)$, and we continue recursively.

($2 \Rightarrow 3$) By Zariski's main theorem, f factors as $Y \xrightarrow{i} Y' \xrightarrow{f'} X$ with i an open immersion and f' a finite morphism. By (2), $Y' \cong \coprod_y \operatorname{Spec} \mathcal{O}_{Y',y}$, where y ranges over finitely many closed points of Y. Hence, we can reduce to the case of a finite étale local homomorphism $A \to B$ such that $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$. Since B is finitely generated as module over A and A is local, it follows that B is a free A-module, and since $\kappa(\mathfrak{m}_B) = B \otimes_A \kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_A)$ it must have rank 1, so $A \cong B$.

($3 \Rightarrow 4$) Let $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ and let $J = \det(\partial f_i/\partial x_j) \in B$. The element $a \in k^n$ defines a morphism $B \to k$, corresponding to a maximal ideal $\mathfrak{q} \subset B$ over $\mathfrak{m}$, such that J is a unit in $B_{\mathfrak{q}}$. Hence, J is a unit in B_b for some $b \notin B \setminus \mathfrak{q}$, and thus B_b is étale over A. The section from (3) corresponds to the desired element $a' \in A^n$.

($4 \Rightarrow 1$) Suppose $\bar{f} = g_0 h_0$. The equation $f = gh$ corresponds to a system of polynomial equations in the coefficients of g and h. There is a solution over k by assumption. From (4) it follows that this solution can be lifted to a solution over A. □

Remark 7. Characterization (3) can be interpreted as: Henselian rings are those for which the 'inverse function theorem' holds.

Corollary 8. *If A is Henselian, so is any finite A-algebra B.*

Proof. Any finite B-algebra is a finite A-algebra, so the result follows from characterization (2). □

Proposition 9. *Any complete local ring A is Henselian.*

Proof. Use characterization (3) of Proposition 6. Let B be an étale A-algebra and $s_0 : B \to k$ a section. To find a lift $s : B \to A = \varprojlim A/\mathfrak{m}^i$ of s_0 it suffices to find compatible lifts $s_i : B \to A/\mathfrak{m}^{i-1}$ for all $i \geq 0$. For $i = 0$ this section is already given, and for $i > 0$ it follows by induction and $A \to B$ being formally étale as indicated in the diagram:

$$\begin{array}{ccc} A/\mathfrak{m}^i & \xleftarrow{s_{i-1}} & B \\ \uparrow & \swarrow_{s_i} & \uparrow \\ A/\mathfrak{m}^{i+1} & \longleftarrow & A \end{array}$$

□

Example 10. Let $f : Y \to X$ be étale and suppose $\kappa(x) = \kappa(y)$ for some $y \in Y$ and $x = f(y)$. The morphism on the completions of stalks $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$ is étale, and since $\hat{\mathcal{O}}_{X,x}$ is Henselian (as it is complete), there is a section. Therefore, it is an isomorphism.

2 Finite étale over Henselian

Proposition 11. *Let A be an Henselian ring with maximal ideal $\mathfrak{m}$ and residue field k. Then the functor $B \mapsto B \otimes_A k$ induces an equivalence of categories*

$$\mathbf{F\acute{E}tAlg}_A \simeq \mathbf{F\acute{E}tAlg}_k$$

between the category of finite étale A-algebras and the category of finite étale k-algebras.

Proof. Let us show the functor is fully faithful and essentially surjective. By Proposition 6 (2), it suffices to restrict our attention only to local finite étale A-algebras. Let B and B' be two local finite étale A-algebras, and consider the map

$$\operatorname{Hom}_A(B, B') \to \operatorname{Hom}_k(B \otimes_A k, B' \otimes_A k). \tag{$*$}$$

To show $(*)$ is surjective, pick any $\varphi : B \otimes_A k \to B' \otimes_A k$ and construct the following commutative diagram

$$\begin{array}{ccc} B' \otimes_A B & \xleftarrow{\ b' \mapsto b' \otimes 1\ } & B' \\ & {\scriptstyle b' \otimes b \mapsto b' \cdot \varphi(b \otimes 1)} \searrow \quad \swarrow {\scriptstyle b' \mapsto b' \otimes 1} & \\ & B' \otimes_A k & \end{array}$$

This defines an étale morphism $f : Y \to X$ from $Y = \operatorname{Spec}(B' \otimes_A B)$ to $X = \operatorname{Spec}(B')$ with a point $y \in Y$ such that $f(y) = x$ and $\kappa(y) = \kappa(x)$. Hence, by Proposition 6 (3) there exists a section $s^\# : B' \otimes_A B \to B'$. Precomposing with $B \to B' \otimes_A B$, $b \mapsto 1 \otimes b$ gives the desired lift $B \to B'$.

To show $(*)$ is injective, let $g, h : B \to B'$ be two A-algebra morphisms such that $g \otimes_A k = h \otimes_A k$. The graphs

$$\Gamma_g, \Gamma_h : B' \otimes_A B \to B', \quad \Gamma_g(b' \otimes b) = b' \cdot g(b), \quad \Gamma_h(b' \otimes b) = b' \cdot h(b)$$

are both sections to f which agree on the closed point. Since f is étale (and separated) these sections must be equal, so $g = h$ [1, Corollary 3.13].

Finally, to see that the functor is essentially surjective, note that any local étale k-algebra k' is of the form $k[x]/(f_0)$ for some $f_0 \in k[x]$ monic and irreducible. Hence, for any lift $f \in A[x]$ of f_0, the finite étale A-algebra $B = A[x]/(f)$ satisfies $B \otimes_A k \cong k'$. ☐

Corollary 12. *If A is Henselian with residue field k, then $\pi_1^{\text{ét}}(\operatorname{Spec} A) \cong \pi_1^{\text{ét}}(\operatorname{Spec} k)$.* ☐

Remark 13. There is a geometric analogue of the above proposition. If X is a scheme proper over a Henselian ring A, and $X_0 = X \times_A k$, then there is an equivalence of categories

$$\mathbf{F\acute{E}t}_X \simeq \mathbf{F\acute{E}t}_{X_0}$$

given by $Y \mapsto Y \times_X X_0$. However, we will omit the proof of this statement.

Corollary 14. *If A is Henselian, there is an equivalence of categories $\mathbf{F\acute{E}tAlg}_A \simeq \mathbf{F\acute{E}tAlg}_{\hat{A}}$.*

3 Henselization

Definition 15. Let A be a local ring. The *Henselization* of A is a local homomorphism of local rings $i : A \to A^{\mathrm{h}}$ such that A^{h} is Henselian and any other local homomorphism from A to a Henselian ring factors through i. Clearly the Henselization of A is unique up to isomorpism, if it exists.

The Henselization of a local ring A can be constructed as the direct limit

$$(A^{\mathrm{h}}, \mathfrak{m}^{\mathrm{h}}) = \varinjlim (B, \mathfrak{q})$$

over the filtered direct system of *étale neighborhoods* $(B, \mathfrak{q})$ of A, that is, B is an étale A-algebra and $\mathfrak{q}$ a prime ideal of B lying over $\mathfrak{m}$ such that $k \to \kappa(\mathfrak{q})$ is an isomorphism. [Stacks 04GN]

Alternatively, if A is noetherian, it can be seen as a subring of its completion $\hat{A}$. Then one can define A^{h} to be the intersection B of all Henselian subrings $A \subset H \subset \hat{A}$ such that $\hat{\mathfrak{m}} \cap H = \mathfrak{m}_H$. Indeed, B is Henselian since any factorization $\overline{f} = g_0 h_0$ lifts to a unique factorization $f = gh$ in any Henselian subring $H \subset \hat{A}$, and hence $g, h \in H[x]$ for all such subrings, and so $g, h \in B[x]$. Therefore, there is a unique local homomorphism $A^{\mathrm{h}} \to B$, whose image is again Henselian and must be equal to B.

Example 16.

- Let X be a scheme and $x \in X$ a point. Then the Henselization of the stalk $\mathcal{O}_{X,x}$ (in the Zariski topology) is given by $\mathcal{O}^{\mathrm{h}}_{X,x} = \varinjlim \Gamma(Y, \mathcal{O}_Y)$, where the limit is taken over all étale neighborhoods of x, which are pairs (Y, y) with Y a connected scheme étale over X and $y \in Y$ a point mapping to x with $\kappa(y) = \kappa(x)$.

- Let A be the localization of $k[x_1, \ldots, x_n]$ at $(x_1, \ldots, x_n)$ for some field k. The Henselization of A is the subring of $k[\![x_1, \ldots, x_n]\!]$ of power series that are algebraic over A. [Corollary 4.17 of Lecture Notes Étale Cohomology Milne]

- Let A be a noetherian local ring, and $I \subset A$ an ideal. Then the Henselization of A/I is $A^{\mathrm{h}}/IA^{\mathrm{h}}$. Indeed, any morphism from A/I to a Henselian ring H corresponds uniquely to a morphism $A \to H$ such that the image of I is zero. This corresponds uniquely to morphism $A^{\mathrm{h}} \to H$ such that the image of IA^{h} is zero, that is, a morphism $A^{\mathrm{h}}/IA^{\mathrm{h}} \to H$, as desired.

- Let $R = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at (p). The Henselization of R is the integral closure of R inside $\mathbb{Z}_p$.

4 Strict Henselization

By characterization (3) of Proposition 6, a Henselian ring A has no non-trivial connected finite étale extensions of A whose residue field extensions is trivial. In particular, if the residue field of A is separably algebraically closed, then A has no connected finite étale extensions at all. Such rings we call *strictly Henselian.*

Definition 17. A Henselian ring A is *strictly Henselian* if its residue field is separably algebraically closed.

Definition 18. Let A be a local ring. The *strict Henselization* of A is a local homomorphism of local rings $i : A \to A^{\mathrm{sh}}$ such that A^{sh} is strictly Henselian and any other local homomorphism $f : A \to H$ to a stricly Henselian ring H extends to a local homomorphism $f' : A^{\mathrm{sh}} \to H$, and moreover f' is to be uniquely determined once the induced map $A^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to H/\mathfrak{m}_H$ on residue fields is given.

The strict Henselization of a local ring A can be constructed as follows. Fix a separable closure k^{sep} of the residue field k of A. Then A^{sh} is given by $\varinjlim B$, where the limit runs over all commutative diagrams:

$$\begin{array}{ccc} B & \longrightarrow & k^{\mathrm{sep}} \\ {\scriptstyle \text{étale}}\uparrow & \nearrow & \\ A & & \end{array}$$

Example 19.

- If $A = k$ is a field, then $A^{\mathrm{h}} = k$ and $A^{\mathrm{sh}} = k^{\mathrm{sep}}$.
- $(A/I)^{\mathrm{sh}} = A^{\mathrm{sh}}/IA^{\mathrm{sh}}$
- Let $\overline{x} \to X$ be a geometric point of X. Then the strict Henselization of the stalk $\mathcal{O}_{X,\overline{x}}$ (in the Zariski topology) is given by $\mathcal{O}^{\mathrm{sh}}_{X,\overline{x}} = \varinjlim \Gamma(U, \mathcal{O}_U)$, where the limit is taken over the étale neighborhoods of $\overline{x}$. This is precisely the stalk of $\mathcal{O}_X$ at $\overline{x}$ in the étale topology.

5 Bonus

Proposition 20. *Let X be a smooth (resp. étale) scheme over a Henselian noetherian ring A with residue field k. Then the map $X(A) \to X(k)$ is surjective (resp. bijective).*

Proof. Locally, we can assume $X = \operatorname{Spec} B$ for some étale algebra $A[x_1, \dots, x_n] \xrightarrow{\varphi} B$-algebra (with $n = 0$ in the étale case). Any k-point of X corresponds to a morphism $B \xrightarrow{f} k$. Let $I \subset A[x_1, \dots, x_n]$ be the ideal generated by $x_i - f(\varphi(x_i))$. Restricting to the corresponding closed subscheme, we obtain an étale algebra $B' = B \otimes_{A[x_1,\dots,x_n]} A[x_1, \dots, x_n]/I$ over $A = A[x_1, \dots, x_n]/I$. Now, the k-point lifts to a unique A-point of $B \otimes_{A[x_1,\dots,x_n]} A$, and hence to an A-point of B, as indicated in the diagram.

$$\begin{array}{ccccc} k & \longleftarrow & B \otimes_{A[x_1,\dots,x_n]} A & \longleftarrow & B \\ \uparrow & \swarrow & \uparrow {\scriptstyle \text{étale}} & & {\scriptstyle \varphi}\uparrow{\scriptstyle \text{étale}} \\ A & = & A & \longleftarrow & A[x_1, \dots, x_n] \end{array}$$

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References

[1] James S. Milne. *Étale cohomology.* Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980, pp. xiii+323.